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MARK SCHEME

Q1

B1 1st term
B1 2d term [redacted]
B1 all ✓

$f'(x) = 2xe^{-x^2} - 2x^3e^{-x^2}$ (AEF)

$f'(x) = 0 \Rightarrow x = 0, 1, -1,$ only — M1A1 — 5

* $P'(x) - 2xP(x) \equiv x(x^2 - a^2)(x^2 - b^2)$ — M1A1 — 2

$P(x)$ of form $-x^4/2 + px^2 + q$ — M2 for a quartic (A1, A1 for zero coeffts) — M2A1A1

$\Rightarrow \ldots 2p + 2 = a^2 + b^2$ and $2p - 2q = a^2b^2$ — M1A1A1

$\Rightarrow \ldots \Rightarrow p = (a^2 + b^2)/2 - 1 \quad q = -1 + (a^2 + b^2)/2 - a^2b^2/2$ — M1A2A2

$P(x) = -x^4/2 + (a^2/2 + b^2/2 - 1)x^2 + -1 + a^2/2 + b^2/2 - a^2b^2/2$ — A1

→ any multiple will do. ✓ — 13

Other methods are marked similarly

* a, b for a^2, b^2 ⇒ M1 A0 here + all other ft if ✓ft MAX 19/20

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MARK SCHEME

Q2

(a) (i) $f(12) = 12(1 - 1/2)(1 - 1/3) = 4$ **M1A1**

$f(180) = 180(1 - 1/2)(1 - 1/3)(1 - 1/5) = 48$ **A1** 3

(ii) $N = p_1^{\alpha_1} \dots p_k^{\alpha_k} \Rightarrow \dots \Rightarrow f(N) = p_1^{\alpha_1 - 1} \dots p_k^{\alpha_k - 1}(p_1 - 1) \dots (p_k - 1)$, **M1A2**

(Not necessary to observe α_i's ≥ 1)

No comment required

Give D2 for partial arguments - do not accept 'obviously true' 3

Could use $f(12)$ $f(18)$ e.g.

(b) (i) 'False': counter example, e.g., $f(3)f(9) = 2 \times 6 = 12 \neq f(27) = 18$ **B1M1A1** 3

TRUE

(ii) p, q prime $\Rightarrow f(p)f(q) = p(1 - 1/p)q(1 - 1/q) = pq(1 - 1/p)(1 - 1/q) = f(pq)$ **B1M1A1** 3

FALSE

(iii) Argument such as the following will earn available marks: B1

Counterexample

~~If p_1 is a prime such that $p_1|p$ and $p_1|q$, then $p_1 - 1|f(p)$ and $p_1 - 1|f(q)$ so that $(p_1 - 1)^2|f(p)f(q)$.~~

Relative prime-ness the key issue here

M1A1

~~$f(pq) \neq (p_1 - 1) \times$ an integer not having $p_1 - 1$ as a factor.~~ ~~**B1**~~ 3

(c) $p^m(1 - 1/p) = 146410$ $= 11^4 \times 10 = 11^5\left(\frac{10}{11}\right) = 11^5\left(1 - \frac{1}{11}\right)$ retrospective to answer. **B1**

[~~and~~ or D1 in failed cases] $\Rightarrow$ MAX 2/5

$p = 11$, $m = 5$ **B2B2**

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MARK SCHEME

Q3

$dy/dx = x\sin x \geq 0$ for $0 \leq x \leq \pi/2$ — B1

$y(0) = 0,\ y(\pi/2) = 1$ — B1

Hence $0 \leq y \leq 1$ for $0 \leq x \leq \pi/2$ → explanation. — B1

Sketch — B2 (D2)

Deduct 1 for each of: incorrect shape at O, incorrect curvature, point $(1, \pi/2)$ on curve not indicated in some way   Allow clearly stated ≡'s

5

(i) $\int_0^{\pi/2} \sin x\, dx = \ldots = 1$ — B1

$\int_0^{\pi} x\cos x\, dx = \pi/2 - 1$ By parts — M1

$\int_0^{\pi/2} y\, dx = 2 - \pi/2$ — A1

3

(ii) $\int_0^{\pi/2} \sin^2 x\, dx = \ldots = \pi/4$ — M1A1

$\int_0^{\pi/2} x\sin 2x\, dx = \ldots = \pi/4$ — M1A1

$\int_0^{\pi/2} x^2\cos^2 x\, dx = \ldots = \pi^3/48 - \pi/8$ — M1 dM1 A1   dble < + 1st parts   2nd parts

$\int_0^{\pi/2} y^2 = \pi/4 - \pi/4 + \pi^3/48 - \pi/8$ (AG) — A1

8

$0 \leq y \leq 1 \Rightarrow y^2 < y$ in $\left(0, \frac{\pi}{2}\right) \Rightarrow \int y^2\,dx < \int y\,dx$ M1A1

3

answer   M1A1

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MARK SCHEME

N.B. no ...

Q4 LHS $= \tan^{-1}($ any correct form in $a, b, c)$ **M1A1**

$= \ldots = \tan^{-1}(1/a)$ (AG) **M1A1** 4

$\tan^{-1}[1/(p+q+s)] + \tan^{-1}[1/(p+q+t)] = \tan^{-1}[1/(p+q)]$ **M1A1**

$\tan^{-1}[1/(p+r+u)] + \tan^{-1}[1/(p+r+v)] = \tan^{-1}[1/(p+r)]$ **M1A1**

$\tan^{-1}[1/(p+q)] + \tan^{-1}[1/(p+r)] = \tan^{-1}(1/p)$ **M1A1** 6

$s = 5$, $t = 13$ **M1A1A1**

$u = 25$, $v = 130$ **M1A1A1**

$p = 7$, $q = 1$, $r = 50$ **M1A1A1A1**

10

Give B1 for each of p, q, r, s, t, u, v, if written down (Max 7/10) other 3/10 for checking the 3 conditions.

$\exists$ 4! solutions depending on the order they choose

$(p+q+s, \ldots) = (13, 21, \ldots)$

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MARK SCHEME

Q5

If S_1 touches sides BC, CA, AB at P, Q, R, respectively, then:

$AQ = AR = r \Rightarrow BR = c - r$, $CQ = b - r$ — M1A1

$\Rightarrow b - r + c - r = a \Rightarrow \ldots \Rightarrow 2r = b + c - a$ — M1A1

4

$r = a(q-1)/2$ — B1

$A_c = \frac{\pi a^2}{4}$ B1

$R = [2bc - \pi a^2(q-1)^2]/\pi a^2$ (AEF) — B1M2A1A1

M2 ratio attempt

$a^2 = b^2 + c^2$ — B1

incorporated

$\Rightarrow 2bc = (a+b+c)(b+c-a) \Rightarrow \ldots \Rightarrow bc/a^2 = (q^2-1)/2$ — M2A1A1

M2 A1 to here

$\pi R = q^2 - 1 - \pi(q-1)^2$ — M1A1

$\Rightarrow \pi R = (1-\pi)q^2 + 2\pi q - 1 - \pi$ (AG) — A1 — 13

NOT THIS FORM

$= -(\pi-1)q^2 + 2\pi q - (\pi+1)$

Shows that $\overset{\pi}{R_{max}} = 1/(\pi - 1)$ (AG) — M1A1A1 — 3

? Diffn. $\frac{dR}{dq} = \text{----}$

but no justification of max

no final A mark

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MARK SCHEME

Q6 (i) x^n, $(n+1)x^n$, $(1/2)(n+1)(n+2)x^n$	Simplified – one B1 for coeffs only (or embedded in a series)	B1B1B1 3
$\Rightarrow \sum_{n=1}^{\infty} n2^{-n} = \ldots = 2$		M1A1 2
$\sum_{n=0}^{\infty} n(n+1)2^{-n} = \cancel{16}\ 8$ (OR EQUIVALENT) B1 as an alt approach	$2S_3 - 3S_2 + S_1$	M1 [redacted] M1 $x=1/2$
$\sum_{n=1}^{\infty} n^2 2^{-n} = 6$		Answer [redacted]A1 3
(ii) General term of $(1-x)^{-1/2}$ (ACF)	unsimplified	B1
Obtain required result (AG)	M for introducing even terms NS. Dr.	M1A1 3
Puts $x = 1/3$ to obtain first series $= \sqrt{3/2}$		M1[redacted]A1 2
Differentiates displayed result in line minus 2 of question		M2A1 Diffn.
Puts $x = 1/3$ to show that sum of second series = [redacted] $\frac{1}{8}\sqrt{6}$		LHS M1A1[redacted] RHS M1 A1 ↳ extra factor of x. 7

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MARK SCHEME

Q7

$\underline{i} - \underline{j}$

(i) (P describes circle centre O and radius 1) (in the $x - y$ plane) **B1B1** 2

$\underline{j}$ and $\sqrt{3}\,\underline{i} + \underline{k}$

(Q describes circle centre O and radius 3) in the plane $\sqrt{3}x - z = 0$ **(B1)M1A1** 3

plane thro' Oy at $\pi/3$ to Ox (y)

(ii) $\cos\theta = |(1/2)\cos t\cos(t + \pi/4) + \sin t\sin(t + \pi/4)|$ **M1A1A1**

$= \ldots = |3/4\sqrt{2} - (1/4)\cos(2t + \pi/4)|$ **M1A1A1** 6

(iii) $\theta \geqslant \pi/4 \Rightarrow 3/4\sqrt{2} - c/4 \leqslant 1/\sqrt{2}$ $\quad (c \equiv \cos(2t + \pi/4)$ **M1A1**

$\Rightarrow -1/\sqrt{2} \leqslant c$ **A1**

$\Rightarrow t \notin [\pi/4, \pi/2]$ and $t \notin [5\pi/4, 3\pi/2]$ **M1A1M1A1**

$\Rightarrow T = 3\pi/2$ **M1A1**

9

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MARK SCHEME

Q8

Use method of separation of variables to obtain:

$A - 1/y = -(1/3)x^2(1+x^2)^{-3/2} - (2/3)(1+x^2)^{-1/2}$ — (M1A1 LHS) M1A1A1 RHS.

$y(0) = 1 \Rightarrow A = 1/3$ — **M1A1**

$1/y = 1/3 + (2+3x^2)/[3(1+x^2)^{3/2}]$ (AG) — **A1** 8

$1/y = 1/3 + 1/x + O(1/x^3)$ — **M1A1**

(D2 for first term .... weaker efforts)

M2 for manipulating into form $k + o(\frac{1}{x})$ as 1st 2 terms
A1 A1 for each.

$\Rightarrow \ldots \Rightarrow y \approx 3 - 9/x$ (AG) — **M1A1** 4

Sketch — Ignore spurious LHS $(x < 0)$ graph bits. (Penalise $\frac{dy}{dx} \neq 0$ at $x = 0$. Asymptote $y = 3$) — **D2** 2

$y = z^2$ leads to second DE — M1 Sep.Vars. M1 A1 $\int$n. etc. — **M2A1**

Sketch — Ditto above. — **D3**

Z_2 = refln Z_1 in x-axis

Asymptotes $z = \pm\sqrt{3}$ identified.

Q9

(i) Diagram (T's not reqd) **B2**

If direction of P makes an angle $\theta + \pi/6$ with horizontal, then

N2L W's F's —

$P\cos\theta = mg + mg/2 + (1/2\sqrt{3})(mg\sqrt{3}/2) + (1/\sqrt{3})(mg\sqrt{3} - P\sin\theta)$ **M1A1A1(A1)**

[P up plane scores max 5/6]

$\Rightarrow \ldots \Rightarrow P\sin(\theta + \pi/3) = (11\sqrt{3}/8)mg$ **M1A1A1**

$P_{min} = (11\sqrt{3}/8)mg$ **M1A1**

$P_{min} = \frac{11}{4}mg$ common from taking $P \nearrow$ along ℓ 5/12 max

P then acts at a direction making an angle of $\pi/3$ with the horizontal **A1**

(ii) $P\cos\theta = mg + mg/2 - (1/2\sqrt{3})(mg\sqrt{3}/2) - (1/\sqrt{3})(mg\sqrt{3} - P\sin\theta)$ **M1A1A1**

(P horizontal M1A1A0 max 2/8)

$\Rightarrow \ldots \Rightarrow P\cos(\theta + \pi/6) = (\sqrt{3}/8)mg$ **M1A1A1**

Minimum of P achieved when $\theta = -\pi/6$, i.e. when P acts horizontally **M1**

In this case, $P_{min} = (\sqrt{3}/8)mg$ **A1**

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MARK SCHEME

Q10

$x_1 = 80t$ — B1

$y_1 = 60t - 5t^2$ — M1A1

$x_2 = 180 - 120(t - T) \quad (t \geq T)$ — B1

$y_2 = 160(t - T) - 5(t - T)^2$ — M1A1

$x_1 = x_2 \Rightarrow 200t = 120T + 180$ (AEF) (H) — M2[redacted]

$y_1 = y_2 \Rightarrow 60t - 5t^2 = 160(t - T) - 5(t - T)^2$ — M2[redacted]

$\Rightarrow \ldots \Rightarrow T^2 + (32 - 2t)T - 20t = 0$ (AEF) (V) — M1A1

(H) and (V) $\Rightarrow T^2 + [(151 - 6T)/5]T - 12T - 18 = 0$ (AEF in T only) — M1A1A1

$T^2 - 91T + 90 = 0$ — A1

$\Rightarrow T = 1, 90$ — solving a quadratic M1A1

Required value of $T = 1$: explanation CWO — B1B1

$T = 90$ gives $t = 54.9 < T$ ie anti-missile missile launched first

or $t = 10\frac{180}{80} < 90$ seconds for 1st missile to reach B.

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MARK SCHEME

Q11

Complete diagram **B2**

Frictional force opposing motion of B is $2m_1g/5$ **B1**

If T is tension on the string and the acceleration is a :

$T - 2m_1g/5 - 3m_1g/5 = m_1a$ **M1A1**

Can do N2L for whole system. i.e. all together

$m_2g - T = m_2a = m_2a$ **A1**

$a = [(m_2 - m_1)/(m_2 + m_1)]g$ (AG) **M1A1** 8

If u, T_1 are the initial velocity and time for the second phase, then

$u = \lambda gT$, $T_1 = \lambda T$, where $\lambda = (m_2 - m_1)/(m_2 + m_1)$ **B1B1**

Total time to highest point is $(1 + \lambda)T$ (AEF) **B1**

For downward motion the key equation is:

$(g/10)(1 + \lambda)^2 = (\lambda g/2)T^2 + (\lambda^2 g/2)T^2$ **B1B1B1M1A1**

$\Rightarrow \ldots \Rightarrow \lambda = 4 \Rightarrow \ldots \Rightarrow m_1/m_2 = 3/5$ **M1A2A1**

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MARK SCHEME

Q12

$T \sim$ true , $F \sim$ false , $\alpha \sim$ heads , $\beta \sim$ tails

(i) $P(T) = ap + bq$	B3
$P(\alpha) = 1/2 \Leftrightarrow ap + bq = 1/2$	M1A1 5
(ii) Write $G = ap + bq$	
$P(\alpha) = P(\alpha TF)/[P(\alpha TF) + P(\beta TF)]$	M2A2
$= [G(1-G)/2]/[G(1-G)/2 + G(1-G)/2] = 1/2$	M1A1A1 7
(iii) $P(\alpha) = P(\alpha TT)/[P(\alpha TT) + P(\beta FF)]$	M2A2
$= [G^2/2]/[G^2/2 + (1-G)^2/2] = 1/2$	A2
$G = 1/2 \Rightarrow ... \Rightarrow P(\alpha) = 1/2$	M1A1 8

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MARK SCHEME

Q13

$X \sim Poi(\lambda \Rightarrow p = P(X < 2) = e^{-\lambda}(1 + \lambda)$ **M1**

$Y \sim Bin(n, p) \Rightarrow P(Y = k) = \binom{n}{k} p^k q^{n-k}$ (AG) Justify Bin. Distn. Success/failure Const. prob. success independent. **A1** 2

(i) $q = 1 - (1 + \lambda)[1 - \lambda + \lambda^2/2 + O(\lambda^3)]$ as $\lambda \to 0$ **M1**

What about ....

$q = e^{-\lambda}\left(\frac{1}{2}\lambda^2 + \frac{1}{3!}\lambda^3 + \cdots\right)$ 2

$= \ldots = \lambda^2/2 + O(\lambda^3) \approx \lambda^2/2$ (AG) $\approx 1 \cdot \frac{1}{2}\lambda^2$ missing $e^{-\lambda} \approx 1 - \lambda$ approxn. $0 + 0\lambda$ **A1** A1 3

(ii) $P(Y = n) = p^n > 1 - \lambda \Rightarrow e^{-n\lambda}(1 + \lambda)^n > 1 - \lambda$ **M1A1**

$\Rightarrow -n\lambda + n\lambda - n\lambda^2/2 > -\lambda - \lambda^2/2 + O(\lambda^3)$ **M2A2**

$\Rightarrow \ldots \Rightarrow n < 2/\lambda$ **A2** 8

(iii) $P(Y > 1|Y > 0) = P(Y > 1)/P(Y > 0) = (1 - q^n - npq^{n-1})/(1 - q^n)$ **M2**

$\approx 1 - np(\lambda^2/2)^{n-1}/[1 - (\lambda^2/2)^n]$ **M1A2**

$\ldots \approx 1 - n(\lambda^2/2)^{n-1}$ (AG) **A2** 7

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MARK SCHEME

Q14

$k = 1/(1 + \lambda)$ **B1**

$E(X) = (0 + \lambda^2/2)/(1 + \lambda) = \lambda^2/[2(1 + \lambda]$ $= \frac{k\lambda^2}{2}$ **M1A1**

$E(X^2) = k + k\lambda^3/3 = (\lambda^3 + 3)/[3(1 + \lambda)]$ **M1A1**

$V(X) = (\lambda^3 + 3)/[3(1 + \lambda)] - \lambda^4/[4(1 + \lambda)^2)]$ **M1**

1 bracket.

$= \ldots = (\lambda^4 + 4\lambda^3 + 12\lambda + 12)/[12(1 + \lambda)^2]$ **A1** 7

$x = 2$ essential

(i) Graph: B1 for each segment: plus B1 if correct overall **B1B1B1B1**

constant gap

Coords on y axis. NO

4

(ii) $F(x) = \Phi(x)/3$ for $x \leq 0$ **B1**

$F(x) = \Phi(x)/3 + x/3$ for $0 < x \leq 2$ **M1A1**

$F(x) = \Phi(x)/3 + 2/3$ for $x > 2$ **A1**

4

(iii) $\mu = \frac{2}{3}$, $\sigma^2 = 7/9$, both **B1**

less $-P(X<0)$

(M0 if $F(x)$ used for $x<2$)

$P(0 < X < \mu + 2\sigma) = .9921/3 + 2/3 = 0.9974$ $-\frac{1}{6}$ **M2A2**

$= 0.8307$

5

(M2 A1 A0 if $F(0)$ missing)